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INVESTIGATION OF STABILITY OF SOLUTIONS OF SOME NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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We study a system of nonlinear differential equations (1). Imposing certain restrictions on the functions appearing on the right in (1) and on the roots of the secular equation, we obtain a number of stability theorems, on stability in the large and on the instability of the zero solution.

Let us consider the following system of differential equations

$$\frac{dx_s}{dt} = p_{s1}\varphi_1(t, x_1, \ldots, x_n) + \ldots + p_{sn}\varphi_n(t, x_1, \ldots, x_n) \qquad (s = 1, 2, \ldots, n) \tag{1}$$

where P_{sk} are real constants, while functions φ_s are defined and continuous in the region given by $(h)\ t \geqslant 0,\ \|x\| = \sqrt{x_1^2 + \ldots + x_n^2} < A, \qquad (\varphi_s(t, 0, \ldots, 0) \equiv 0)$

In certain isolated cases we can base our deductions about the stability or instability of the zero solution of (1) on the properties of the roots of

$$\det \| p_{sk} - \lambda\delta_{sk} \| = 0 \tag{2}$$

For example, we can formulate the following theorems.

Theorem 1. Let the right sides of (1) be such that the function

$$u(t, x_1, \ldots, x_n) = \sum_{s=1}^{n} a_s\varphi_s(t, x_1, \ldots, x_n) \tag{3}$$

where at least one of the numbers $a_s \neq 0$, is sign-definite. If, at the same time, Eq. (2) has no zero roots, then the zero solution of (1) is unstable.

Proof. We shall seek the following linear form

$$v(x_1, \ldots, x_n) = \sum_{s=1}^{n} b_s x_s \tag{4}$$

whose total derivative satisfies, by (1), the relation

$$v' = \sum_{s=1}^{n} b_s [p_{s1}\varphi_1(t, x_1, \ldots, x_n) + \ldots + p_{sn}\varphi_n(t, x_1, \ldots, x_n)] =$$
$$= \sum_{s=1}^{n} a_s\varphi_s(t, x_1, \ldots x_n) = u(t, x_1, \ldots, x_n) \quad (5)$$

Equating the coefficients of $\varphi_s(t, x_1, \ldots, x_n)$ on both sides of (5), we obtain the system

$$p_{1s}b_1 + \ldots + p_{ns}b_n = a_s \quad (s = 1, 2, \ldots, n)$$

whose determinant is not equal to zero.

Thus the required form v exists and, as it can easily be seen, satisfies all conditions of the first Liapunov theorem on instability.

Theorem 2. Let the coefficients of system (1) satisfy the condition

$$p_{sk} \geqslant 0 \quad \text{when } s \neq k \qquad (s, k = 1, 2, \ldots, n) \quad (6)$$

and let us suppose that such positive constants α_s exist, that the function

$$U(t, x_1, \ldots, x_n) = \sum_{s=1}^{n} \alpha_s\varphi_s(t, x_1, \ldots, x_n)\operatorname{sgn} x_s \quad (7)$$

(where $\varphi_s(t, x_1, \ldots, x_n) = 0$ when $x_s = 0$), is always positive. If, at the same time, all roots of Eq. (2) have negative real parts, then the zero solution of the system under consideration is stable.

Proof. We consider the following linear system of differential equations:

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n \qquad (s = 1, 2, \ldots, n) \quad (8)$$

whose coefficients p_{sk} coincide with the corresponding coefficients of (1).

From Lemma 4.1 given by Krasnosel'skii in [1] it follows that if condition (6) holds, then

$$(G)\ t \geqslant 0,\ x_s \geqslant 0 \qquad (s = 1, 2, \ldots, n)$$

constitutes a positive invariant set for system (8).

Let us choose the following linear form:

$$v(x_1, \ldots, x_n) = \sum_{s=1}^{n} \beta_s x_s \quad (9)$$

whose total derivative satisfies, by virtue of (8), the following relation in the region G:

$$v' = \sum_{s=1}^{n} \beta_s(p_{s1}x_1 + \ldots + p_{sn}x_n) = -\sum_{s=1}^{n} \alpha_s x_s \quad (10)$$

Such a choice of v is feasible, since the determinant of

$$p_{1s}\beta_1 + \ldots + p_{ns}\beta_s = -\alpha_s \quad (s = 1, 2, \ldots, n) \quad (11)$$

is not equal to zero. In addition, all β_s defined by (11) will be positive.

Indeed, using (10) we can easily show that all $\beta_s \neq 0$; moreover, if only one $\beta_s < 0$, then the form (9) will satisfy in G all conditions of the first Liapunov theorem on instability, by virtue of the system (8), and the zero solution of this system is asymptotically stable. Thus, all $\beta_s > 0$.

Let us now put

$$V(x_1, \ldots, x_n) = \sum_{s=1}^{n} \beta_s |x_s| \quad (12)$$

Using (6) and (11) we easily find that the right total derivative of V and G will satisfy, by virtue of (1), the inequality

$$V' \leqslant \sum_{s=1}^{n} (\beta_1 p_{1s} + \ldots + \beta_n p_{ns}) \varphi_s (t, x_1, \ldots, x_n) \operatorname{sgn} x_s =$$
$$= - \sum_{s=1}^{n} \alpha_s \varphi_s (t, x_1, \ldots, x_n) \operatorname{sgn} x_s = - U (t, x_1, \ldots, x_n).$$

This implies that all conditions of the Liapunov theorem on stability hold for the system (1).

The following theorem is proved in an entirely analogous manner.

Theorem 3. If conditions of Theorem 2 hold and the function $U(t, x_1, \ldots, x_n)$ defined by (7) is positive-definite for some $\alpha_s > 0$, then the zero solution of (1) is asymptotically stable uniformly in t_0 and x_{s0}.

Corollary. Let the domain h of definition of the right sides of (1) be given by $t \geqslant 0$, $\| x \| < \infty$.

If conditions of the Theorem 3 hold in this domain, then the zero solution of the system considered is asymptotically stable in the large uniformly in t_0 and x_{s0}.

Indeed, in this case, by virtue of the system (1), form (12) will satisfy all conditions of the theorem on the asymptotic uniform stability in the large given in [2].

Let us further consider the system of equations

$$\frac{dx_s}{dt} = p_{s1}\varphi_1 (x_1, \ldots, x_n) + \ldots + p_{sn}\varphi_n (x_1, \ldots, x_n) + R_s (x_1, \ldots, x_n)$$
$$(s = 1, 2, \ldots, n) \tag{13}$$

where p_{sk} are real constants, φ_s denote certain polynomials in $x_1, \ldots, x_n$ of degree not larger than m, while the functions R_s can be expanded in series in powers of x_s in some neighborhood of the origin. These series begin with terms of order not less than $m + 1$.

Theorem 4. Let the system (13) be such, that the function

$$u (x_1, \ldots, x_n) = \sum_{s=1}^{n} a_s \varphi_s (x_1, \ldots, x_n) \tag{14}$$

where at least one of the numbers $a_s \neq 0$, is sign-definite. If, in addition, Eq. (2) has no zero roots, then the zero solution of the system under consideration will be unstable.

Indeed, by (13), in this case the total derivative of linear form (4) in a sufficiently small neighborhood of the coordinate origin will be a sign-definite function.

Theorem 5. Let the coefficients p_{sk} of (13) satisfy (6). If, in addition, the function

$$u (x_1, \ldots, x_n) = \sum_{s=1}^{n} \alpha_s \varphi_s (x_1, \ldots, x_n) \operatorname{sgn} x_s \quad (a_0 > 0) \tag{15}$$

where $\varphi_s (x_1, \ldots, x_n) = 0$ when $x_s = 0$ is positive-definite and if all the roots of (2) have negative real parts, then the zero solution of the system under consideration is asymptotically stable uniformly in t_0 and x_{s0}. To conclude, we consider several examples. Let the following system of equations be given:

$$\begin{aligned} x' &= -2xy + R_1 (x, y, z) \\ y' &= 2yz + y^2 + R_2 (x, y, z) \\ z' &= zy - 2z^2 - x^2 + R_3 (x, y, z) \end{aligned} \tag{16}$$

Here the expansions of the functions R_s into series in the powers of x, y and z begin in some neighborhood of the origin with terms of order not lower than the third. Setting

$$\varphi_1 = 2xy, \ \varphi_2 = 2yz + y^2, \ \varphi_3 = zy - 2z^2 - x^2$$

and choosing $\alpha_1 = 0$, $\alpha_2 = -1$ and $\alpha_3 = 2$ we find that the function

$$u(x, y, z) = \sum_{s=1}^{3} \alpha_s \varphi_s = -2x^2 - y^2 - 4z^2$$

becomes negative-definite, and that Eq. (2) of the system has no zero roots. Therefore, by Theorem 4, the zero solution of (16) is unstable.

We further consider the system

$$\frac{dx_s}{dt} = p_{s1}\varphi_1(x_1) + \ldots + p_{sn}\varphi_n(x_n) \qquad (s = 1, 2, \ldots, n) \tag{17}$$

where p_{sk} are real constants and $\varphi_s(x_s)$ are continuous functions satisfying the conditions

$$\varphi_s(0) = 0; \quad \varphi_s(x_s) \operatorname{sgn} x_s > 0 \text{ when } x_s \neq 0 \quad (s = 1, 2, \ldots, n) \tag{18}$$

If $\varphi_k(x_k) = x_k$ when $k = 1, \ldots, l$ where $l < n$, we obtain a control system with one or more control mechanisms.

Let the coefficients of (17) satisfy (6), and let all roots of (2) have negative real parts. Then the corollary of Theorem 3 implies that the zero solution of the system will be uniformly asymptotically stable in the large for any continuous functions $\varphi_s(x_s)$ satisfying (18).

Note. If the above assumptions concerning the right-hand sides of the system (17) hold and if Eq. (2) has no zero roots but at least one of its roots has a positive real part, then the zero solution of the system is unstable.

Indeed, when conditions (6) and (18) hold, then the region G will be [1] a positive invariant set for system (17). It can easily be shown that a function v of the type (4) whose coefficients b_s are to be determined from the system

$$p_{1s}b_1 + \ldots + p_{ns}b_n = 1 \quad (s = 1, 2, \ldots, n) \tag{19}$$

will satisfy in G (by virtue of (17)) all conditions of the first Liapunov theorem on instability.

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